Introduction to Mathematics with Proofs

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Overview

1. Proofs
2. Mathematical Sets (not Python!)

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4. Because my parents said it's true! No, you SUCK!

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- Proofs leverage facts or things we reasonably assume to be true to make conclusions about more facts...
- in the language of **mathematics**

Example

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Prove that if a number is greater than 6, then it **must** be greater than 5.

Proof.

Well.. it's obvious? Proof by obviousness! □

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Proof.

Let x be an arbitrary number that is greater than 6.

$$x > 6 > 5$$

which shows the "then" part, as wanted. □

BAD Example

Example

Prove that Yufei is the most handsome.

BAD Example

Example

Prove that Yufei is the most handsome.

Proof.

Linda looked at Yufei and compared it against other people around the room and has confirmed it. □

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3. In order to do so, we need to learn some math nomenclature – the building blocks of the mathematical language.
4. Then we will try to "communicate" in that language.

Outline

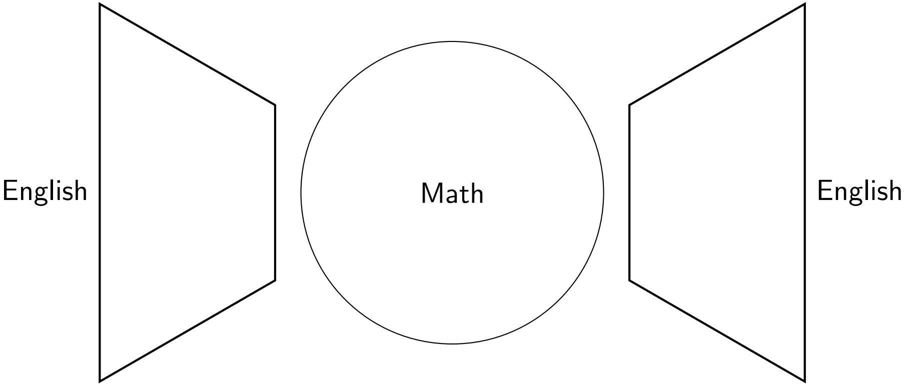

Figure: Mathematical Maturity

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- But for our purposes, we will accept this as it intuitively make sense.

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Example

$$A = \{1, 2, 3\}$$

$$B = \{"apple", "pear", "oranges"\}$$

$$C = \{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}$$

$$D = \{\text{all Canadians with 4 as the last digit of their SIN card}\}$$

We use "{}" to denote a set and the things inside are its "members".

Notations on Sets

Now that we have introduced the mathematical object of a set, we can now play around with sets.

Notations on Sets

$$X = \{1, 4, 5\} \quad Y = \{4, 7, 10\} \quad Z = \{1, 5\}$$

Symbol	**Meaning**	**Example**
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$\cap$	"intersection of"	$X \cap Y = \{4\}$
$\varnothing$	"empty set"	$Y \cap Z = \varnothing$

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Note:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Set-Building Notation

$$\underbrace{A}_{\text{Name of the set}} = \{ \overbrace{x \in \mathbb{N}}^{\text{where you're taking elements from}} \underbrace{:}_{\text{such that}} \overbrace{x^2 < 4}^{additional constraints} \}$$

Set-Building Notation

Practice: Explicitly write the elements of

1. $E = \{x \in \mathbb{Z} : x^2 < 4\}$
2. $F = \{y \in \mathbb{N} : -17 < x < 10\}$
3. $G = E \cap F$
4. $H = G \cup F$